

# Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media

Buyang Li and Weiwei Sun\*

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## Abstract

In this paper, we study the unconditional convergence and error estimates of a Galerkin-mixed FEM with the linearized semi-implicit Euler time-discrete scheme for the equations of incompressible miscible flow in porous media. We prove that the optimal  $L^2$  error estimates hold without any time-step (convergence) condition, while all previous works require certain time-step condition. Our theoretical results provide a new understanding on commonly-used linearized schemes. The proof is based on a splitting of the error function into two parts: the error from the time discretization of the PDEs and the error from the finite element discretization of corresponding time-discrete PDEs. The approach used in this paper is applicable for more general nonlinear parabolic systems and many other linearized (semi)-implicit time discretizations.

## 1 Introduction

Incompressible miscible flow in porous media is governed in general by the following system of equations:

$$\Phi \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}) \nabla c) + \mathbf{u} \cdot \nabla c = \hat{c} q^I - c q^P, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = q^I - q^P, \quad (1.2)$$

$$\mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla p, \quad (1.3)$$

where  $p$  is the pressure of the fluid mixture,  $\mathbf{u}$  is the velocity and  $c$  is the concentration of the first component;  $k(x)$  is the permeability of the medium,  $\mu(c)$  is the concentration-dependent viscosity,  $\Phi$  is the porosity of the medium,  $q^I$  and  $q^P$  are

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\*Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong. The work of the authors was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 102005)

Email address: buyangli2@student.cityu.edu.hk, maweiw@math.cityu.edu.hk

given injection and production sources,  $\hat{c}$  is the concentration of the first component in the injection source, and  $D(\mathbf{u}) = [D_{ij}(\mathbf{u})]_{d \times d}$  is the diffusion-dispersion tensor which may be given in different forms (see [4, 5] for details). We assume that the system is defined in a bounded smooth domain  $\Omega$  in  $\mathbb{R}^d$  ( $d = 2, 3$ ), for  $t \in [0, T]$ , coupled with the initial and boundary conditions:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0, \quad D(\mathbf{u}) \nabla c \cdot \mathbf{n} = 0 \quad \text{for } x \in \partial\Omega, \quad t \in [0, T], \\ c(x, 0) &= c_0(x) \quad \text{for } x \in \Omega. \end{aligned} \tag{1.4}$$

The above system was investigated extensively in the last several decades [3, 9, 17, 18, 20] due to its wide applications in various engineering areas, such as reservoir simulations and exploration of underground water, oil and gas. Existence of weak solutions of the system was obtained by Feng [20] for the 2D model and by Chen and Ewing [6] for the 3D problem. Existence of semi-classical/classical solutions is unknown so far. Numerical simulations have been done extensively with various applications, see [10, 12, 29] and the references therein. Optimal error estimates of a standard Galerkin-Galerkin method for the system (1.1)-(1.4) in two-dimensional space was obtained first by Ewing and Wheeler [19] roughly under the time-step condition  $\tau = o(h)$ . In their method, a linearized semi-implicit Euler scheme was used in the time direction and Galerkin FEM approximation was used both for the concentration and the pressure. Later, a Galerkin-mixed finite element method was proposed by Douglas et al [11] for this system, where a Galerkin approximation was applied for the concentration equation and a mixed approximation in the Raviart-Thomas finite element space was used proposed for the pressure equation. A linearized semi-implicit Euler scheme, the same as that used in [19], was applied for the time discretization. Optimal error estimates were obtained under a similar time-step condition  $\tau = o(h)$ . In [28], the authors studied the system in two-dimensional space by the linearized backward Euler scheme and a streamline-upwind-Petrov-Galerkin method combined with a post-process technique on the velocity. Error estimates with quasi-optimal rates were derived. There are many other numerical methods in literature for solving the equations of incompressible miscible flow, such as [35, 38] with an ELLAM in two-dimensional space, [36] with an MMOC-MFEM approximation for the 2D problem, [33, 13] with a characteristic-mixed method in two and three dimensional spaces, respectively, and [26, 27] with a collocation-mixed method and a characteristic-collocation method, respectively. In all those works, error estimates were established under certain time-step conditions. Moreover, linearized semi-implicit schemes have also been analyzed for many other nonlinear parabolic-type systems, among which many require certain time-step restrictions when strong nonlinearities were involved, see [1, 2, 9, 8, 14, 15, 21, 22, 24, 25, 37, 38] and the references therein. Such time-step restrictions may result in the use of a very small time step and extremely time-consuming in practical computations. The problem becomes more serious when a non-uniform mesh is used. However, we believe that those time-step restrictions are not necessary in most cases.

In this paper, we analyze the linearized semi-implicit Euler scheme with a popular Galerkin-mixed finite element approximation in the spatial direction for the system (1.1)-(1.4). We establish optimal  $L^2$  error estimates almost without any

time-step restriction (or when  $h$  and  $\tau$  are smaller than some positive constants, respectively). This provides a new understanding on the commonly-used linearized schemes and clears up the misgivings for the time-step size in practical computations. Our theoretical analysis is based on an error splitting proposed in [23] for a Joule heating system with a standard Galerkin FEM. By introducing a corresponding time-discrete system, we split the numerical error into two parts, the error in the spatial direction and the error in the temporal direction. The rigorous analysis of the regularity of the solution to the time-discrete PDEs is a key to our approach because of the strong coupling of the system. With such proved regularity, we obtain optimal and  $\tau$ -independent  $L^2$ -error estimates of the Galerkin-mixed FEM for the time-discrete PDEs.

The rest of the paper is organized as follows. In Section 2, we introduce the linearized semi-implicit Euler scheme with a Galerkin-mixed approximation in the spatial direction for the system (1.1)-(1.4) and present our main results. In Section 3, we present a priori estimates of the solution to the corresponding time-discrete system and the error estimate of the linearized scheme. Optimal error estimates of the fully discrete scheme in the  $L^2$  norm are given in Section 4. Conclusions are drawn in the last section. Since numerical simulations on the underlying system have been done extensively, we will not provide any numerical results in this paper.

## 2 The Galerkin-mixed FEM and the main results

For any integer  $m \geq 0$  and  $1 \leq p \leq \infty$ , let  $W^{m,p}$  be the Sobolev space of functions defined on  $\Omega$  equipped with the norm

$$\|f\|_{W^{m,p}} = \left( \sum_{|\beta| \leq m} \int_{\Omega} |D^{\beta} f|^p dx \right)^{\frac{1}{p}},$$

where

$$D^{\beta} = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}$$

for the multi-index  $\beta = (\beta_1, \dots, \beta_d)$ ,  $\beta_1 \geq 0, \dots, \beta_d \geq 0$ , and  $|\beta| = \beta_1 + \dots + \beta_d$ . For any integer  $m \geq 0$  and  $0 < \alpha < 1$ , let  $C^{m+\alpha}$  denote the usual Hölder space equipped with the norm

$$\|f\|_{C^{m+\alpha}} = \sum_{|\beta| \leq m} \|D^{\beta} f\|_{C(\overline{\Omega})} + \sum_{|\beta|=k} \sup_{x,y \in \Omega} \frac{|D^{\beta} f(x) - D^{\beta} f(y)|}{|x - y|^{\alpha}}.$$

Let  $I = (0, T)$ . For any Banach space  $X$ , we consider functions  $g : I \rightarrow X$  and define the norm

$$\|g\|_{L^p(I; X)} = \begin{cases} \left( \int_0^T \|g(t)\|_X^p dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in I} \|g(t)\|_X, & p = \infty. \end{cases}$$

In addition, we define  $L_0^p$  as the subspace of  $L_p$  consisting of functions in  $L_p$  whose integral over  $\Omega$  are zeros. Finally, we denote by  $H$  the space of vector-valued functions  $\vec{f} \in L^2 \times L^2 \times L^2$  such that  $\nabla \cdot \vec{f} \in L^2$ .

Let  $\pi_h$  be a regular subdivision of  $\Omega$  into triangles  $T_j$ ,  $j = 1, \dots, M$ , in  $\mathbb{R}^2$  (or tetrahedra in  $\mathbb{R}^3$ ), with  $\Omega_h = \cup_j T_j$  and denote by  $h = \max_{1 \leq j \leq M} \{\text{diam } T_j\}$  the mesh size. For a triangle  $T_j$  at the boundary, we define  $\tilde{T}_j$  as the extension of  $T_j$  to the triangle with one curved edge (or a tetrahedra with one curved face in  $\mathbb{R}^3$ ). For a given subdivision of  $\Omega$ , we define the finite element spaces:

$$\begin{aligned} S_h &= \{w_h \in L^2(\Omega) : w_h|_{\tilde{T}_j} \text{ is linear for each element } T_j \in \pi_h\}, \\ V_h &= \{w_h \in C^0(\overline{\Omega}) : w_h|_{\tilde{T}_j} \text{ is linear for each element } T_j \in \pi_h\}. \end{aligned}$$

Let  $H_h$  be the subspace of  $H$ , as introduced by Raviart and Thomas [31, 34] such that  $\text{div } \psi \in S_h$  for  $\psi \in H_h$ .

In the rest part of this paper, we assume that the solution to the initial-boundary value problem (1.1)-(1.4) exists and satisfies

$$\begin{aligned} &\|p\|_{L^\infty(I; H^3)} + \|\mathbf{u}\|_{L^\infty(I; H^2)} + \|\mathbf{u}_t\|_{L^2(I; W^{1,3/2})} + \|c\|_{L^\infty(I; W^{2,s})} \\ &+ \|c_t\|_{L^\infty(I; L^s)} + \|c_t\|_{L^s(I; W^{1,s})} + \|c_{tt}\|_{L^s(I; W^{-1,s})} \leq C \end{aligned} \quad (2.1)$$

for some  $s > d$  and

$$\|q^I\|_{H^1}, \|q^P\|_{H^1} \leq C. \quad (2.2)$$

Correspondingly, we assume that the permeability  $k \in C^2(\overline{\Omega})$  and satisfies

$$k_0^{-1} \leq k(x) \leq k_0 \quad \text{for } x \in \Omega, \quad (2.3)$$

the concentration-dependent viscosity  $\mu \in C^1(\mathbb{R})$  and satisfies

$$\mu_0^{-1} \leq \mu(s) \leq \mu_0 \quad \text{for } s \in \mathbb{R}, \quad (2.4)$$

for some positive constant  $\mu_0$ . Moreover, we assume that the diffusion-dispersion tensor is given by  $D(\mathbf{u}) = \Phi d_m I + D^*(\mathbf{u})$ , where  $d_m > 0$ ,  $D^*(\mathbf{u}) = d_1(\mathbf{u})I + d_2(\mathbf{u})(\mathbf{u} \otimes \mathbf{u})$  is symmetric and positive definite and  $\partial_{u_i u_j}^2 D \in L^\infty$  [5]. For the initial-boundary value problem (1.1)-(1.4) to be well-posed, we require

$$\int_{\Omega} q^I dx = \int_{\Omega} q^P dx. \quad (2.5)$$

Let  $\{t_n\}_{n=0}^N$  be a uniform partition of the time interval  $[0, T]$  with  $\tau = T/N$  and denote

$$p^n = p(x, t_n), \quad \mathbf{u}^n = \mathbf{u}(x, t_n), \quad c^n = c(x, t_n).$$

For any sequence of functions  $\{f^n\}_{n=0}^N$ , we define

$$D_t f^{n+1} = \frac{f^{n+1} - f^n}{\tau}.$$

The fully discrete mixed finite element scheme is to find  $P_h^n \in S_h/\{\text{constant}\}$ ,  $U_h^n \in H_h$  and  $\mathcal{C}_h^n \in V_h$ ,  $n = 0, 1, \dots, N$ , such that for all  $v_h \in H_h$ ,  $\varphi_h \in S_h$  and  $\phi_h \in V_h$ ,

$$\left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1}, v_h \right) = - \left( P_h^{n+1}, \nabla \cdot v_h \right), \quad (2.6)$$

$$\left( \nabla \cdot U_h^{n+1}, \varphi_h \right) = \left( q^I - q^P, \varphi_h \right), \quad (2.7)$$

$$\begin{aligned} \left( \Phi D_t \mathcal{C}_h^{n+1}, \phi_h \right) + \left( D(U_h^{n+1}) \nabla \mathcal{C}_h^{n+1}, \nabla \phi_h \right) \\ + \left( U_h^{n+1} \cdot \nabla \mathcal{C}_h^{n+1}, \phi_h \right) = \left( \hat{c} q^I - \mathcal{C}_h^{n+1} q^P, \phi_h \right) \end{aligned} \quad (2.8)$$

where the initial data  $\mathcal{C}_h^0$  is the Lagrangian piecewise linear interpolation of  $c^0$ .

In this paper, we denote by  $C$  a generic positive constant and by  $\epsilon$  a generic small positive constant, which are independent of  $n$ ,  $h$  and  $\tau$ . We present our main results in the following theorem.

**Theorem 2.1** *Suppose that the initial-boundary value problem (1.1)-(1.4) has a unique solution  $(p, \mathbf{u}, c)$  which satisfies (2.1). Then there exist positive constants  $h_0$  and  $\tau_0$  such that when  $h < h_0$  and  $\tau < \tau_0$ , the finite element system (2.6)-(2.8) admits a unique solution  $(P_h^n, U_h^n, \mathcal{C}_h^n)$ ,  $n = 1, \dots, N$ , which satisfies that*

$$\max_{1 \leq n \leq N} \|P_h^n - p^n\|_{L^2} + \max_{1 \leq n \leq N} \|U_h^n - \mathbf{u}^n\|_{L^2} + \max_{1 \leq n \leq N} \|\mathcal{C}_h^n - c^n\|_{L^2} \leq C(\tau + h^2). \quad (2.9)$$

We will present the proof of Theorem 2.1 in the next two sections. The key to our proof is the following error splitting

$$\begin{aligned} \|U_h^n - \mathbf{u}^n\|_{L^2} &\leq \|e_u^n\|_{L^2} + \|U^n - U_h^n\|_{L^2}, \\ \|P_h^n - p^n\|_{L^2} &\leq \|e_p^n\|_{L^2} + \|P^n - P_h^n\|_{L^2}, \\ \|\mathcal{C}_h^n - c^n\|_{L^2} &\leq \|e_c^n\|_{L^2} + \|\mathcal{C}^n - \mathcal{C}_h^n\|_{L^2}, \end{aligned}$$

where

$$\begin{aligned} e_p^n &= P^n - p^n, \\ e_u^n &= U^n - \mathbf{u}^n, \\ e_c^n &= \mathcal{C}^n - c^n, \end{aligned}$$

and  $(P^n, U^n, \mathcal{C}^n)$  is the solution of a time-discrete system defined in next section.

### 3 Error estimates for time-discrete system

We define the time-discrete solution  $(P^n, U^n, \mathcal{C}^n)$  by the following elliptic system:

$$U^{n+1} = - \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla P^{n+1}, \quad (3.1)$$

$$\nabla \cdot U^{n+1} = q^I - q^P, \quad (3.2)$$

$$\Phi D_t \mathcal{C}^{n+1} - \nabla \cdot (D(U^{n+1}) \nabla \mathcal{C}^{n+1}) + U^{n+1} \cdot \nabla \mathcal{C}^{n+1} = \hat{c} q^I - \mathcal{C}^{n+1} q^P, \quad (3.3)$$

for  $x \in \Omega$  and  $t \in [0, T]$ , with the initial and boundary conditions

$$\begin{aligned} U^{n+1} \cdot \mathbf{n} &= 0, \quad D(U^{n+1}) \nabla \mathcal{C}^{n+1} \cdot \mathbf{n} = 0 \quad \text{for } x \in \partial\Omega, \quad t \in [0, T], \\ \mathcal{C}^0(x) &= c_0(x) \quad \text{for } x \in \Omega, \end{aligned} \quad (3.4)$$

In this section, we prove the existence and uniqueness and suitable regularity of the solution of the above time-discrete system.

**Theorem 3.1** *Suppose that the initial-boundary value problem (1.1)-(1.4) has a unique solution  $(p, \mathbf{u}, c)$  which satisfies (2.1). Then there exists a positive constant  $\tau_0$  such that when  $\tau < \tau_0$ , the time-discrete system (3.1)-(3.4) admits a unique solution  $(P^n, U^n, \mathcal{C}^n)$ ,  $n = 1, \dots, N$ , which satisfies*

$$\|P^n\|_{H^2} + \|U^n\|_{H^2} + \|\mathcal{C}^n\|_{W^{2,s}} + \|D_t \mathcal{C}^n\|_{L^s} + \|\nabla \mathcal{C}^n\|_{L^\infty} + \left( \sum_{n=1}^{N-1} \tau \|\nabla D_t U^{n+1}\|_{L^{3/2}}^2 \right)^{\frac{1}{2}} \leq C, \quad (3.5)$$

and

$$\max_{1 \leq n \leq N} \|e_p^n\|_{L^s} + \max_{1 \leq n \leq N} \|e_u^n\|_{L^s} + \max_{1 \leq n \leq N} \|e_c^n\|_{L^s} + \left( \sum_{n=1}^N \tau \|\nabla e_c^n\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C\tau. \quad (3.6)$$

*Proof.* It suffices to establish the estimates presented in (3.5). With such estimates, existence and uniqueness of solution follow a routine way.

We observe that  $e_p^n$ ,  $e_u^n$  and  $e_c^n$  satisfy the following equations

$$-\nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^{n+1} \right) = \nabla \cdot \left[ \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^{n+1} \right], \quad (3.7)$$

$$e_u^{n+1} = -\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^{n+1} - \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^{n+1}, \quad (3.8)$$

$$\begin{aligned} \Phi D_t e_c^{n+1} - \nabla \cdot (D(U^{n+1}) \nabla e_c^{n+1}) + U^n \cdot \nabla e_c^{n+1} \\ = \nabla \cdot \left( (D(U^{n+1}) - D(\mathbf{u}^{n+1})) \nabla c^{n+1} \right) - (U^n - \mathbf{u}^n) \cdot \nabla c^{n+1} - e_c^{n+1} q^P + \mathcal{E}^{n+1}, \end{aligned} \quad (3.9)$$

for  $x \in \Omega$  and  $t \in [0, T]$ , with the initial and boundary conditions

$$\begin{aligned} \frac{k(x)}{\mu(\mathcal{C}^{n+1})} \nabla e_p^{n+1} \cdot \mathbf{n} &= 0, \quad D(U^{n+1}) \nabla e_c^{n+1} \cdot \mathbf{n} = 0, \quad \text{for } x \in \partial\Omega, \quad t \in [0, T], \\ e_c^0(x) &= 0, \quad \text{for } x \in \Omega, \end{aligned} \quad (3.10)$$

where

$$\mathcal{E}^{n+1} = \Phi(c_t^{n+1} - D_t c^{n+1})$$

is the truncation error due to the discretization in the time direction. From the regularity assumption for  $c$  in (2.1), we see that

$$\|\mathcal{E}^{n+1}\|_{L^2} \leq C, \quad \sum_{n=0}^{N-1} \tau \|\mathcal{E}^{n+1}\|_{H^{-1}}^2 \leq C\tau^2. \quad (3.11)$$

To prove the error estimate (3.6), we multiply (3.7) by  $e_p^{n+1}$  to get

$$\|\nabla e_p^{n+1}\|_{L^2} \leq \left\| \left( \frac{k(x)}{\mu(C^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^{n+1} \right\|_{L^2} \leq C \|e_c^n\|_{L^2} \|\nabla p^{n+1}\|_{L^\infty} \leq C \|e_c^n\|_{L^2} \quad (3.12)$$

which together with (3.8) implies that

$$\|e_u^{n+1}\|_{L^2} \leq C \|\nabla e_p^{n+1}\|_{L^2} + C \|e_c^{n+1}\|_{L^2} \|\nabla p^{n+1}\|_{L^\infty} \leq C \|e_c^n\|_{L^2}. \quad (3.13)$$

Then we multiply (3.9) by  $e_c^{n+1}$  to get

$$\begin{aligned} & \frac{1}{2} D_t \left( \Phi \|e_c^{n+1}\|_{L^2}^2 \right) + \|\sqrt{D(U^{n+1})} \nabla e_c^{n+1}\|_{L^2}^2 \\ & \leq C \|e^{n+1}\|_{L^4}^2 \|q^I - q^P\|_{L^2} + C \|e_u^n\|_{L^2} \|\nabla e_c^{n+1}\|_{L^2} \|\nabla c^{n+1}\|_{L^\infty} \\ & \quad + C \|e_u^n\|_{L^2} \|e_c^{n+1}\|_{L^2} \|\nabla c^{n+1}\|_{L^\infty} + C \|e_c^{n+1}\|_{L^4}^2 \|q^P\|_{L^2} + C \|\mathcal{E}^{n+1}\|_{H^{-1}} \|e_c^{n+1}\|_{H^1} \\ & \leq C \|e_c^{n+1}\|_{L^4}^2 + C \|e_c^{n+1}\|_{H^1} (\|e_c^{n+1}\|_{L^2} + \|e_c^n\|_{L^2}) + C \|\mathcal{E}^{n+1}\|_{H^{-1}}^2 \\ & \leq \epsilon \|\nabla e_c^{n+1}\|_{L^2}^2 + C_\epsilon \|e_c^{n+1}\|_{L^2}^2 + C \|e_c^n\|_{L^2}^2 + C \|\mathcal{E}^{n+1}\|_{H^{-1}}^2, \end{aligned} \quad (3.14)$$

where we have used the inequalities

$$|(U^{n+1} \cdot \nabla e_c^{n+1}, e_c^{n+1})| = |(\nabla \cdot U^{n+1}, |e_c^{n+1}|^2)| \leq \|e^{n+1}\|_{L^4}^2 \|q^I - q^P\|_{L^2},$$

and

$$\|e_c^{n+1}\|_{L^4}^2 \leq \epsilon \|\nabla e_c^{n+1}\|_{L^2}^2 + C_\epsilon \|e_c^{n+1}\|_{L^2}^2.$$

The square root of  $D(U^{n+1})$  exists because  $D(U^{n+1})$  is a symmetric and positive definite matrix. With Gronwall's inequality and (3.11), (3.14) reduces to

$$\|e_c^{n+1}\|_{L^2}^2 + \sum_{n=0}^{N-1} \tau \|\nabla e_c^{n+1}\|_{L^2}^2 \leq C\tau^2, \quad (3.15)$$

which further produces

$$\|D_t e_c^{n+1}\|_{L^2} \leq C.$$

From (3.12)-(3.13), we derive that

$$\|e_u^{n+1}\|_{L^2} + \|\nabla e_p^{n+1}\|_{L^2} \leq C\tau. \quad (3.16)$$

To prove (3.5), we rewrite (3.9) as

$$\begin{aligned}
& \Phi D_t e_c^{n+1} - \nabla \cdot (D(\mathbf{u}^{n+1}) \nabla e_c^{n+1}) \\
&= -\nabla \cdot \left( (D(\mathbf{u}^{n+1}) - D(U^{n+1})) \nabla e_c^{n+1} \right) - (U^n - \mathbf{u}^n) \cdot \nabla e_c^{n+1} - \mathbf{u}^n \cdot \nabla e_c^{n+1} \\
&\quad + \nabla \cdot \left( (D(U^{n+1}) - D(\mathbf{u}^{n+1})) \nabla e_c^{n+1} \right) - (U^n - \mathbf{u}^n) \cdot \nabla e_c^{n+1} - e_c^{n+1} q^P + \mathcal{E}^{n+1}.
\end{aligned} \tag{3.17}$$

Since  $\mathbf{u}^n \in C^\alpha(\bar{\Omega})$ , using the  $W^{1,6}$  estimates of elliptic equations for  $e_c$  [7], we get

$$\begin{aligned}
\|\nabla e_c^{n+1}\|_{L^6} &\leq C \|D_t e_c^{n+1}\|_{L^2} + C \|e_u^{n+1}\|_{L^\infty} \|\nabla e_c^{n+1}\|_{L^6} + C \|\mathbf{u}^n\|_{L^\infty} \|\nabla e_c^{n+1}\|_{L^2} \\
&\quad + C \|e_u^{n+1}\|_{L^\infty} \|\nabla e_c^{n+1}\|_{L^6} + C \|q^P\|_{L^6} \|e_c^{n+1}\|_{L^3} + C \|\mathcal{E}^{n+1}\|_{L^2}
\end{aligned}$$

which further reduces to

$$\|\nabla e_c^{n+1}\|_{L^6} \leq C_0 \|e_u^{n+1}\|_{L^\infty} (C_0 + \|\nabla e_c^{n+1}\|_{L^6}) + C_0 \tag{3.18}$$

for some positive constant  $C_0$  independent of  $n, \tau, h$ .

On the other hand, we rewrite the equation (3.7) into the following form:

$$-\frac{k(x)}{\mu(\mathcal{C}^n)} \Delta e_p^{n+1} = \nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \cdot \nabla e_p^{n+1} \right) + \nabla \cdot \left[ \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^{n+1} \right]. \tag{3.19}$$

Since we have

$$\begin{aligned}
\left\| \nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \cdot \nabla e_p^{n+1} \right) \right\|_{L^6} &\leq \left\| \nabla \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \right) \right\|_{L^6} \|\nabla e_p^{n+1}\|_{L^\infty} \\
&\leq (C + C \|\nabla \mathcal{C}^n\|_{L^6}) \|\nabla e_p^{n+1}\|_{L^\infty}
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \nabla \cdot \left[ \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^{n+1} \right] \right\|_{L^6} \\
&\leq \left\| \nabla \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \cdot \nabla p^{n+1} \right\|_{L^6} + \left\| \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla \cdot \nabla p^{n+1} \right\|_{L^6} \\
&\leq \|\nabla e_c^n\|_{L^6} \|\nabla p^{n+1}\|_{L^\infty} + \|e_c^n\|_{L^\infty} \|p^{n+1}\|_{W^{2,6}},
\end{aligned}$$

by the  $W^{2,6}$  estimates of elliptic equations [7], we derive from (3.19) that

$$\begin{aligned}
\|e_p^{n+1}\|_{W^{2,6}} &\leq (C + C \|\nabla \mathcal{C}^n\|_{L^6}) \|\nabla e_p^{n+1}\|_{L^\infty} + C \|\nabla e_c^n\|_{L^6} + C \|e_c^n\|_{L^\infty} \\
&\leq (C + C \|\nabla e_c^n\|_{L^6}) (\epsilon \|e_p^{n+1}\|_{W^{2,6}} + C_\epsilon \|e_p\|_{L^2}) \\
&\quad + (C + \epsilon) \|\nabla e_c^n\|_{L^6} + (C + C_\epsilon) \|e_c^n\|_{L^2} \\
&\leq (C + C \|\nabla e_c^n\|_{L^6}) \epsilon \|e_p^{n+1}\|_{W^{2,6}} + C_\epsilon + C_\epsilon \|\nabla e_c^n\|_{L^6}
\end{aligned} \tag{3.20}$$

From (3.8) we observe that

$$\|e_u^{n+1}\|_{W^{1,6}} \leq (C + C \|e_c^n\|_{W^{1,6}}) (\|\nabla e_p^{n+1}\|_{L^\infty} + \|\nabla p^{n+1}\|_{L^\infty}) + C \|e_p^{n+1}\|_{W^{2,6}}$$



$$\leq (C + C\|\nabla e_c^n\|_{L^6})\epsilon\|e_p^{n+1}\|_{W^{2,6}} + C_\epsilon + C_\epsilon\|\nabla e_c^n\|_{L^6}, \quad (3.21)$$

and by the Sobolev interpolation inequality, we have

$$\|e_u^{n+1}\|_{L^\infty} \leq C\|e_u^{n+1}\|_{L^2}^{1/4}\|e_u^{n+1}\|_{W^{1,6}}^{3/4} \leq C\tau^{1/4}\|e_u^{n+1}\|_{W^{1,6}}^{3/4}. \quad (3.22)$$

With the estimates (3.18), (3.20), (3.21) and (3.22), we now apply mathematical induction to prove

$$\|\nabla e_c^n\|_{L^6} \leq 4C_0 \quad (3.23)$$

for  $n = 1, 2, \dots, N$ . Clearly, the above inequality holds when  $n = 0$ . With the above inequalities, by choosing a proper small  $\epsilon = \epsilon(C_0)$ , we derive from (3.20)-(3.22) that

$$\|e_u^{n+1}\|_{L^\infty} \leq C_1\tau^{1/4}, \quad (3.24)$$

where  $C_1$  may depend on  $C_0$ . Hence, there exists a positive constant  $\tau_0 = \tau_0(C_1)$  such that when  $\tau < \tau_0$ , we have

$$C_0\|e_u^{n+1}\|_{L^\infty} \leq 1/2.$$

Substituting the above inequality into (3.18) gives

$$\|\nabla e_c^{n+1}\|_{L^6} \leq 4C_0.$$

By mathematical induction, (3.23) holds for  $1 \leq n \leq N$ . From (3.20) and (3.21), we also get

$$\max_{1 \leq n \leq N} \|e_p^{n+1}\|_{W^{2,6}} + \max_{1 \leq n \leq N} \|e_u^{n+1}\|_{W^{1,6}} \leq C. \quad (3.25)$$

By a similar approach, we have further

$$\begin{aligned} \|\nabla D_t U^{n+1}\|_{L^{3/2}} &\leq \|\nabla D_t e_u^{n+1}\|_{L^{3/2}} + \|\nabla D_t \mathbf{u}^{n+1}\|_{L^{3/2}} \\ &\leq C\tau^{-1}\|\nabla e_u^{n+1}\|_{L^{3/2}} + C\tau^{-1}\|\nabla e_u^n\|_{L^{3/2}} + \|\nabla D_t \mathbf{u}^{n+1}\|_{L^{3/2}} \\ &\leq C\tau^{-1}\|\nabla e_c^n\|_{L^{3/2}} + C\tau^{-1}\|\nabla e_c^{n-1}\|_{L^{3/2}} + \|\nabla D_t \mathbf{u}^{n+1}\|_{L^{3/2}}, \end{aligned}$$

and therefore

$$\left( \sum_{n=1}^{N-1} \tau \|\nabla D_t U^{n+1}\|_{L^{3/2}}^2 \right)^{\frac{1}{2}} \leq C. \quad (3.26)$$

With the above estimates, by applying the  $H^3$  estimates of elliptic equations [16], we can derive that

$$\|e_p^{n+1}\|_{H^3} + \|e_u^{n+1}\|_{H^2} \leq C. \quad (3.27)$$

For the  $H^2$  regularity of  $\mathcal{C}^{n+1}$ , we rewrite (3.3) as

$$-\nabla \cdot (D(U^{n+1})\nabla \mathcal{C}^{n+1}) + U^{n+1} \cdot \nabla \mathcal{C}^{n+1} = -\Phi D_t \mathcal{C}^{n+1} + \hat{c}q^I - \mathcal{C}^{n+1}q^P. \quad (3.28)$$

With  $\|D_t \mathcal{C}^{n+1}\|_{L^2} \leq C$ , we can apply the  $H^2$  estimates for the elliptic equation [16] to obtain

$$\|\mathcal{C}^{n+1}\|_{H^2} \leq C. \quad (3.29)$$

Note that  $H^2 \hookrightarrow C^\alpha$  for some  $\alpha > 0$ . With the Hölder regularity of  $\mathcal{C}^n$ , applying the  $W^{1,s}$  estimates to (3.7) and using (3.8), it is not difficult to see that

$$\|e_u^{n+1}\|_{L^s} \leq C \|e_c^n\|_{L^s}. \quad (3.30)$$

Multiplying (3.9) by  $|e_c^{n+1}|^{s-2} e_c^{n+1}$  and using (3.30), one can derive that

$$\begin{aligned} & \int_{\Omega} s^{-1} \Phi D_t |e_c^{n+1}|^s dx + (s-1) \int_{\Omega} |e_c^{n+1}|^{s-2} D(U^{n+1}) \nabla e_c^{n+1} \cdot \nabla e_c^{n+1} dx \\ & \leq \int_{\Omega} \int_{\Omega} s^{-1} (q^I - q^P) |e_c^{n+1}|^s dx + \|\nabla c^{n+1}\|_{L^\infty} \|U^n - \mathbf{u}^n\|_{L^s} \|e_c^{n+1}\|_{L^s}^{s-1} \\ & \quad + \|q^P\|_{L^\infty} \|e_c^{n+1}\|_{L^s} + \|\mathcal{E}^{n+1}\|_{W^{-1,s}} \|e_c^{n+1}\|_{L^s}^{\frac{s-2}{2}} \| |e_c^{n+1}|^{s-2} \nabla e_c^{n+1} \|_{L^2} \\ & \quad + (s-1) \|D(U^{n+1}) - D(\mathbf{u}^{n+1})\|_{L^s} \|\nabla c^{n+1}\|_{L^\infty} \|e_c^{n+1}\|_{L^s}^{\frac{s-2}{2}} \| |e_c^{n+1}|^{s-2} \nabla e_c^{n+1} \|_{L^2} \\ & \leq (C + \|\mathcal{E}^{n+1}\|_{W^{-1,s}}^s) \epsilon^{-1} (\|e_c^{n+1}\|_{L^s}^s + \|e_c^n\|_{L^s}^s) + \epsilon (s-1) \int_{\Omega} |e_c^{n+1}|^{s-2} |\nabla e_c^{n+1}|^2 dx. \end{aligned}$$

Choosing a proper small  $\epsilon$  and using Gronwall's inequality lead to

$$\|e_c^{n+1}\|_{L^s} \leq C\tau,$$

where we have noted the fact that

$$\|\mathcal{E}^{n+1}\|_{L^s} \leq C, \quad \sum_{n=0}^{N-1} \tau \|\mathcal{E}^{n+1}\|_{W^{-1,s}}^s \leq C\tau^s. \quad (3.31)$$

It follows that

$$\|D_t e_c^{n+1}\|_{L^s} \leq C. \quad (3.32)$$

With the above estimate, a refinement of the above arguments shows that

$$\|e_c^{n+1}\|_{W^{2,s}} \leq C \quad (3.33)$$

and by the Sobolev embedding theorem,  $\|\nabla e_c^{n+1}\|_{L^\infty} \leq C \|e_c^{n+1}\|_{W^{2,s}} \leq C$ .

Finally, by applying the  $W^{1,s}$  estimates to the elliptic equation (3.7) and using (3.8), we obtain

$$\|\nabla e_p^{n+1}\|_{L^s} + \|e_u^{n+1}\|_{L^s} \leq C\tau. \quad (3.34)$$

The proof of Theorem 3.1 is complete. ■

## 4 Error estimates of the fully-discrete system

To provide optimal error estimates for the fully discrete scheme (2.6)-(2.8), we define three projections below.

Let  $\Pi_h : L^2(\Omega) \rightarrow S_h$  be a projection defined in [34] by

$$(\Pi_h w, \chi) = (w, \chi), \quad \forall \chi \in S_h.$$

For any fixed integer  $n > 0$ , let  $\Pi_h^n : H^1(\Omega) \rightarrow V_h$  be a projection defined by the following elliptic problem,

$$\left( D(U^n) \nabla(v - \Pi_h^n v), \nabla \phi_h \right) = 0, \quad \text{for all } \phi_h \in V_h, \quad v \in H^1(\Omega) \quad (4.1)$$

and  $\Pi_h^0 = \Pi_h^1$ . Moreover, let  $Q_h : H \rightarrow H_h$  be a projection defined in [34] by

$$\left( \nabla \cdot (w - Q_h w), \varphi_h \right) = 0, \quad \text{for all } \varphi_h \in S_h, \quad w \in H. \quad (4.2)$$

By the theory of Galerkin and mixed finite element methods for linear elliptic problems [30, 34], with the regularity  $U^n \in H^2(\Omega)$ , we have

$$\begin{aligned} \|v - \Pi_h v\|_{L^2} + h\|v - \Pi_h v\|_{H^1} &\leq Ch^2|v|_{H^2}, & \text{for all } v \in H^2(\Omega), \\ \|v - \Pi_h^{n+1} v\|_{L^p} + h\|v - \Pi_h^{n+1} v\|_{W^{1,p}} &\leq Ch^2|v|_{W^{2,p}}, & \text{for all } v \in W^{2,p}(\Omega), \\ \|w - Q_h w\|_{L^2} + h\|w - Q_h w\|_{H^1} &\leq Ch^2|w|_{H^2}, & \text{for all } w \in H, \end{aligned} \quad (4.3)$$

Previous works on Galerkin (or mixed) FEM for the nonlinear parabolic system (1.1)-(1.3) required the estimate

$$\|\partial_t(c^{n+1} - \tilde{\Pi}_h^{n+1} c^{n+1})\|_{L^2} \leq Ch^2 \quad (4.4)$$

for an elliptic projection operator  $\tilde{\Pi}_h^{n+1}$  defined by

$$\left( D(\mathbf{u}) \nabla(v - \tilde{\Pi}_h v), \nabla \phi_h \right) = 0, \quad \text{for all } \phi_h \in V_h, \quad v \in H^1(\Omega). \quad (4.5)$$

The inequality (4.4) was proved by Wheeler [39] based on the regularity assumption  $\|\nabla D(\mathbf{u})_t\|_{L^\infty} \leq C$ . In the following lemma, we will prove an analogous inequality:

$$\left( \sum_{n=0}^{N-1} \tau \|D_t(\mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{H^{-1}}^2 \right)^{1/2} \leq Ch^2, \quad (4.6)$$

based on weaker regularity of  $U^n$  proved in the last section. The above inequality is necessary to obtain optimal  $L^2$  error estimates.

**Lemma 4.1** *With the regularity assumption (2.1), the regularity of  $(\mathcal{C}^{n+1}, U^{n+1})$  given in (3.5), and the error estimates given in (3.6), the estimate (4.6) holds.*

*Proof* We only prove the lemma for the 3D problem (with  $s > 3$  in Theorem 3.1). The 2D problem can be handled similarly. Note that

$$\left( D(U^{n+1}) \nabla (\mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}), \nabla \phi_h \right) = 0, \quad (4.7)$$

$$\left( D(U^n) \nabla (\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}), \nabla \phi_h \right) = 0. \quad (4.8)$$

The difference of the above two equations gives

$$\begin{aligned} & \left( D(U^{n+1}) \nabla (\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}), \nabla \phi_h \right) \\ & + \left( (D(U^{n+1}) - D(U^n)) \nabla (\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}), \nabla \phi_h \right) = 0. \end{aligned}$$

By the  $W^{1,p}$  estimates of elliptic projections [30], we have

$$\begin{aligned} & \|\nabla (\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^{6/5}} \leq C \|(D(U^{n+1}) - D(U^n)) \nabla (\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1})\|_{L^{6/5}} \\ & \leq C \|D(U^{n+1}) - D(U^n)\|_{L^2} \|\nabla (\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1})\|_{L^3} \leq C \|D(U^{n+1}) - D(U^n)\|_{L^2} h. \end{aligned}$$

For any  $\varphi \in H^1(\Omega)$  with  $\int_\Omega \varphi \, dx = 0$ , let  $\psi$  be the solution of the equation

$$-\nabla \cdot (D(U^{n+1}) \nabla \psi) = \varphi$$

with the boundary condition  $\nabla \psi \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . Since  $U^{n+1}$  is uniformly bounded in  $H^2(\Omega)$ , i.e.  $\|U^{n+1}\|_{H^2} \leq C$ , it is easy to check that

$$\|\psi\|_{H^3} \leq C \|\varphi\|_{H^1}.$$

By the boundary condition  $U^{n+1} \cdot \mathbf{n} = U^n \cdot \mathbf{n} = 0$  and the expression of the function  $D(\mathbf{u})$ , we see that  $D(U^{n+1}) \nabla \psi \cdot \mathbf{n} = D(U^n) \nabla \psi \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . We see that for  $n > 0$ ,

$$\begin{aligned} & (\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}, \varphi) \\ & = \left( D(U^{n+1}) \nabla (\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}), \nabla \psi \right) \\ & = \left( D(U^{n+1}) \nabla (\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}), \nabla (\psi - P_h \psi) \right) \\ & \quad - \left( (D(U^{n+1}) - D(U^n)) \nabla (\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}), \nabla (P_h \psi - \psi) \right) \\ & \quad - \left( (D(U^{n+1}) - D(U^n)) \nabla (\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}), \nabla \psi \right) \\ & \leq C \|D(U^{n+1}) - D(U^n)\|_{L^2} h \|\nabla (\psi - P_h \psi)\|_{L^6} \\ & \quad + \|\mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}\|_{L^3} \left\| \nabla \cdot \left( (D(U^{n+1}) - D(U^n)) \nabla \psi \right) \right\|_{L^{3/2}} \\ & \leq C \|D(U^{n+1}) - D(U^n)\|_{L^2} h^2 \|\psi\|_{W^{2,6}} \\ & \quad + Ch^2 \|\mathcal{C}^{n+1}\|_{W^{2,3}} \left( \|\nabla (D(U^{n+1}) - D(U^n))\|_{L^{3/2}} \|\nabla \psi\|_{L^\infty} + \|U^{n+1} - U^n\|_{L^2} \|\psi\|_{W^{2,6}} \right) \\ & \leq C (\|D_t U^{n+1}\|_{L^2} + C \|\nabla D_t U^{n+1}\|_{L^{3/2}} + \|\nabla U^{n+1} D_t U^{n+1}\|_{L^{3/2}}) \tau h^2 \|\psi\|_{H^3} \end{aligned}$$

$$\leq C (\|D_t U^{n+1}\|_{L^2} + C\|\nabla D_t U^{n+1}\|_{L^{3/2}} + \|\nabla U^{n+1} D_t U^{n+1}\|_{L^{3/2}}) \tau h^2 \|\varphi\|_{H^1}.$$

Therefore,

$$\begin{aligned} & \|\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{H^{-1}} \\ & \leq C (\|D_t U^{n+1}\|_{L^2} + C\|\nabla D_t U^{n+1}\|_{L^{3/2}} + \|\nabla U^{n+1} D_t U^{n+1}\|_{L^{3/2}}) \tau h^2. \end{aligned}$$

By (3.6), we have

$$\begin{aligned} \|D_t U^{n+1}\|_{L^2} & \leq \|D_t e_u^{n+1}\|_{L^2} + \|D_t \mathbf{u}^{n+1}\|_{L^2} \leq C\tau + \|D_t \mathbf{u}^{n+1}\|_{L^2}, \\ \|\nabla U^{n+1} D_t U^{n+1}\|_{L^{3/2}} & \leq \|\nabla U^{n+1}\|_{L^6} \|D_t U^{n+1}\|_{L^2} \leq C\tau + \|D_t \mathbf{u}^{n+1}\|_{L^2}. \end{aligned}$$

With the regularity assumption (2.1) on  $\mathbf{u}$  and the estimate (3.6), we derive that

$$\left( \sum_{n=0}^{N-1} \tau \|\Pi_h^n \mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{H^{-1}}^2 \right)^{\frac{1}{2}} \leq C\tau h^2.$$

Since

$$\begin{aligned} & \|D_t (\mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{H^{-1}} \\ & \leq \|D_t \mathcal{C}^{n+1} - \Pi_h^n D_t \mathcal{C}^{n+1}\|_{H^{-1}} + \tau^{-1} \|\Pi_h^{n+1} \mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}\|_{H^{-1}} \\ & \leq C \|D_t \mathcal{C}^{n+1}\|_{H^1} h^2 + \tau^{-1} \|\Pi_h^{n+1} \mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}\|_{H^{-1}} \\ & \leq C \|D_t e_c^{n+1}\|_{H^1} h^2 + C \|D_t \mathcal{C}^{n+1}\|_{H^1} h^2 + \tau^{-1} \|\Pi_h^{n+1} \mathcal{C}^{n+1} - \Pi_h^n \mathcal{C}^{n+1}\|_{H^{-1}}, \end{aligned}$$

with the estimate (3.6) and the regularity assumption (2.1), (4.6) follows immediately. The proof of Lemma 4.1 is complete. ■

**Theorem 4.1** *Suppose that the initial-boundary value problem (1.1)-(1.4) has a unique solution  $(p, \mathbf{u}, c)$  which satisfies (2.1). Then there exist positive constants  $h_0$  and  $\tau_0$  such that when  $h < h_0$  and  $\tau < \tau_0$ , the finite element system (2.6)-(2.7) admits a unique solution  $(P_h^n, U_h^n, \mathcal{C}_h^n)$ ,  $n = 1, \dots, N$ , which satisfies*

$$\max_{1 \leq n \leq N} \|P_h^n - P^n\|_{L_0^2} + \max_{1 \leq n \leq N} \|U_h^n - U^n\|_{L^2} + \max_{1 \leq n \leq N} \|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^2} \leq Ch^2.$$

where  $(P^n, U^n, \mathcal{C}^n)$ ,  $n = 1, \dots, N$ , are the solution of the time-discrete system (3.1)-(3.4).

*Proof.* The solution to the time-discrete system (3.1)-(3.4) satisfies

$$\left( \frac{\mu(\mathcal{C}^n)}{k(x)} U^{n+1}, v_h \right) = - \left( P^{n+1}, \nabla \cdot v_h \right), \quad (4.9)$$

$$\left( \nabla \cdot U^{n+1}, \varphi_h \right) = \left( q^I - q^P, \varphi_h \right), \quad (4.10)$$

$$\left( \Phi D_t \mathcal{C}^{n+1}, \phi_h \right) + \left( D(U^{n+1}) \nabla \mathcal{C}^{n+1}, \nabla \phi_h \right)$$

$$+ \left( U^n \cdot \nabla \mathcal{C}^{n+1}, \phi_h \right) = \left( \hat{c} q^I - \mathcal{C}^{n+1} q^P, \phi_h \right). \quad (4.11)$$

for any  $v_h \in H_h$ ,  $\varphi_h \in S_h$  and  $\phi_h \in V_h$ . The above equations with the finite element system (2.6)-(2.8) imply that the error functions  $P_h^{n+1} - \Pi_h P^{n+1}$ ,  $U_h^{n+1} - U^{n+1}$ ,  $\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}$  satisfy

$$\left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1} - \frac{\mu(\mathcal{C}^n)}{k(x)} U^{n+1}, v_h \right) = - \left( P_h^{n+1} - \Pi_h P^{n+1}, \nabla \cdot v_h \right), \quad (4.12)$$

$$\left( \nabla \cdot (U_h^{n+1} - U^{n+1}), \varphi_h \right) = 0, \quad (4.13)$$

$$\begin{aligned} & \left( \Phi D_t(\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}), \phi_h \right) + \left( D(U_h^{n+1}) \nabla(\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}), \nabla \phi_h \right) \\ &= - \left( U^{n+1} \cdot \nabla(\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}), \phi_h \right) - \left( (U_h^{n+1} - U^{n+1}) \cdot \nabla \mathcal{C}_h^{n+1}, \phi_h \right) \\ & \quad - \left( (\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}) q^P, \phi_h \right) + \left( (D(U^{n+1}) - D(U_h^{n+1})) \nabla \Pi_h^{n+1} \mathcal{C}^{n+1}, \nabla \phi_h \right). \end{aligned} \quad (4.14)$$

First we present a uniform estimate for the finite element solution. Let  $g$  be the solution of the elliptic problem

$$-\nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) = q^I - q^P$$

with the boundary condition  $\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \cdot \mathbf{n} = 0$  on  $\partial\Omega$ .

Taking  $v_h = Q_h \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right)$  in (2.6) and by noting the definition of the projection  $Q_h$ , we get (with  $v_h \cdot \mathbf{n} = \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \cdot \mathbf{n} = 0$  on  $\partial\Omega$  [34])

$$\begin{aligned} \left( P_h^{n+1}, q^P - q^I \right) &= \left( P_h^{n+1}, \nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) \right) \\ &= \left( P_h^{n+1}, \nabla \cdot v_h \right) \\ &= \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1}, -v_h \right) \\ &\leq C \|U_h^{n+1}\|_{L^2} \|v_h\|_{L^2} \\ &\leq C \|U_h^{n+1}\|_{L^2} \|q^I - q^P\|_{L^2}. \end{aligned}$$

and taking  $v_h = U_h^{n+1}$  in (2.6) and  $\psi_h = P_h^{n+1}$  in (2.7) gives

$$\|U_h^{n+1}\|_{L^2}^2 = - \left( P_h^{n+1}, \nabla \cdot U_h^{n+1} \right) = - \left( P_h^{n+1}, q^I - q^P \right).$$

Combining the last two inequalities, we obtain

$$\|U_h^{n+1}\|_{L^2} \leq C \|q^I - q^P\|_{L^2}. \quad (4.15)$$

Secondly, we present an upper bound for the error function  $\|U_h^{n+1} - U^{n+1}\|_{L^2}$  in terms of  $\|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2}$ . By the definition of the projection operator  $Q_h$  in (4.2),

$$\left( \nabla \cdot (U_h^{n+1} - Q_h U^{n+1}), \varphi_h \right) = 0, \quad \text{for all } \varphi_h \in S_h,$$

which together with (4.13) shows that  $\nabla \cdot (U_h^{n+1} - Q_h U^{n+1}) = 0$  in  $\Omega$ . Taking  $v_h = U_h^{n+1} - Q_h U^{n+1}$  in (4.12), we get

$$\begin{aligned} & \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} (U_h^{n+1} - Q_h U^{n+1}) + \frac{\mu(\mathcal{C}_h^n)}{k(x)} (Q_h U^{n+1} - U^{n+1}) \right. \\ & \quad \left. + \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} - \frac{\mu(\mathcal{C}^n)}{k(x)} \right) U^{n+1}, U_h^{n+1} - Q_h U^{n+1} \right) = 0, \end{aligned}$$

which further implies that

$$\|U_h^{n+1} - Q_h U^{n+1}\|_{L^2} \leq C \|Q_h U^{n+1} - U^{n+1}\|_{L^2} + C \|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^2}.$$

With (4.3), the above inequality reduces to

$$\|U_h^{n+1} - U^{n+1}\|_{L^2} \leq Ch^2 + C \|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^2}. \quad (4.16)$$

Thirdly, we take  $\phi_h = \mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1} - \int_{\Omega} (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) dx$  (so that  $\int_{\Omega} \phi_h dx = 0$ ) in (4.14) and get

$$\begin{aligned} & \frac{1}{2} D_t \left\| \sqrt{\Phi} (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) \right\|_{L^2}^2 + \left\| \sqrt{D(U_h^{n+1})} \nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) \right\|_{L^2}^2 \\ & \leq \epsilon \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2}^2 + \|D_t (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{H^{-1}}^2 \\ & \quad + C \|q^I - q^P\|_{L^3} \|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^6} \|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2} \\ & \quad + C \|U^n\|_{L^\infty} \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2} \|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2} \\ & \quad + C \|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^6} (\|U_h^{n+1} - U^{n+1}\|_{L^2} \|\nabla (\Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^3} \\ & \quad + \|U_h^{n+1} - U^{n+1}\|_{L^3} \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2}) \\ & \quad + C \|q^P\|_{L^3} (\|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^3}^2 + \|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^6} \|\mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^2}) \\ & \quad + C \|\nabla \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^\infty} \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2} \|U_h^{n+1} - U^{n+1}\|_{L^2} \\ & \leq \epsilon \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2}^2 + \|D_t (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{H^{-1}}^2 \\ & \quad + Ch^{-1/2} \|U_h^{n+1} - U^{n+1}\|_{L^2} \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2}^2 \\ & \quad + C \|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^2}^2 + C \|\mathcal{C}_h^n - \Pi_h^n \mathcal{C}^n\|_{L^2}^2 + Ch^4, \end{aligned} \quad (4.17)$$

where we have used (4.3), (4.16) and the following estimate:

$$\begin{aligned} & \left| (U^{n+1} \cdot \nabla (\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}), \mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) \right| \\ & = \left| ((q^I - q^P)(\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}), \mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) \right| \\ & \quad + \left| (U^{n+1} (\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}), \nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})) \right| \\ & \leq C \|q^I - q^P\|_{L^3} \|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^6} \|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2} \\ & \quad + C \|U^{n+1}\|_{L^\infty} \|\nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2} \|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2}. \end{aligned}$$

From (4.16) we observe that (4.17) reduces to

$$D_t \left\| \sqrt{\Phi} (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) \right\|_{L^2}^2 + \left\| \sqrt{D(U_h^{n+1})} \nabla (\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}) \right\|_{L^2}^2$$

$$\begin{aligned} &\leq Ch^{-1/2} \|\mathcal{C}_h^n - \Pi_h^n \mathcal{C}^n\|_{L^2} \|\nabla(\mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{L^2}^2 \\ &+ C \|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^2}^2 + C \|\mathcal{C}_h^n - \Pi_h^n \mathcal{C}^n\|_{L^2}^2 + \|D_t(\mathcal{C}^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1})\|_{H^{-1}}^2 + Ch^4. \end{aligned}$$

By applying Gronwall's inequality with induction, we deduce that there exist  $h_0 > 0$  such that for  $h < h_0$ ,

$$\|\mathcal{C}_h^{n+1} - \Pi_h^{n+1} \mathcal{C}^{n+1}\|_{L^2} \leq Ch^2. \quad (4.18)$$

By (4.16), we get further

$$\|U_h^{n+1} - U^{n+1}\|_{L^2} \leq Ch^2, \quad (4.19)$$

$$\|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2} \leq Ch^2. \quad (4.20)$$

Finally, we estimate the error  $\|P_h - P^{n+1}\|_{L^2}$ . We redefine  $g$  to be the solution to the equation

$$-\nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) = P_h^{n+1} - \Pi_h P^{n+1}$$

with the boundary condition  $\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . Easy to check that

$$\|g\|_{H^2} \leq C \|P_h^{n+1} - \Pi_h P^{n+1}\|_{L^2}.$$

Let

$$v_h = Q_h \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right)$$

Then

$$(\varphi, \nabla \cdot v_h) = -(\varphi, P_h^{n+1} - \Pi_h P^{n+1}), \quad \varphi \in S_h$$

and from (4.12) we obtain

$$\begin{aligned} \|P_h^{n+1} - \Pi_h P^{n+1}\|_{L^2}^2 &= \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1} - \frac{\mu(\mathcal{C}^n)}{k(x)} U^{n+1}, Q_h \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) \right) \\ &= \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1} - \frac{\mu(\mathcal{C}^n)}{k(x)} U^{n+1}, (Q_h - I) \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) \right) \\ &\quad + \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1} - \frac{\mu(\mathcal{C}^n)}{k(x)} U^{n+1}, \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) \\ &= \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^{n+1} - \frac{\mu(\mathcal{C}^n)}{k(x)} U^{n+1}, (Q_h - I) \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right) \right) \\ &\quad + \left( \nabla \cdot (U_h^{n+1} - U^{n+1}), g - \Pi_h g \right) \\ &\quad + \left( \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} - \frac{\mu(\mathcal{C}^n)}{k(x)} \right) U_h^{n+1}, \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla \Pi_h g \right), \end{aligned}$$

where we have used (4.13). By (4.16), we have further

$$\|P_h^{n+1} - \Pi_h P^{n+1}\|_{L^2}^2 \leq Ch (\|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\| + \|U_h^{n+1} - U^{n+1}\|_{L^2}) \left\| \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla g \right\|_{H^1}$$



$$+Ch^2\|\nabla \cdot (U_h^{n+1} - U^{n+1})\|_{L^2}\|g\|_{H^2} + C\|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^2}\|U_h^{n+1}\|_{L^3}\|\nabla g\|_{L^6}.$$

Since

$$\begin{aligned}\|\nabla g\|_{L^6} &\leq C\|g\|_{H^2} \leq C\|P_h^{n+1} - \Pi_h P^{n+1}\|_{L^2}, \\ \|\nabla \cdot (U_h^{n+1} - U^{n+1})\|_{L^2} &\leq C\|U^{n+1}\|_{H^1},\end{aligned}$$

and

$$\left\| \frac{k(x)}{\mu(\mathcal{C}^{n+1})} \nabla g \right\|_{H^1} \leq \|g\|_{H^2} + \|\nabla \mathcal{C}^{n+1}\|_{L^3}\|g\|_{H^2} \leq C\|P_h^{n+1} - \Pi_h P^{n+1}\|_{L^2},$$

it follows immediately that

$$\|P_h^{n+1} - \Pi_h P^{n+1}\|_{L^2} \leq Ch^2.$$

The proof of Theorem 4.1 is complete. ■

Combining Theorems 3.1 and Theorem 4.1, we complete the proof of Theorem 2.1. ■

## 5 Conclusions

We have studied error analysis for a nonlinear and strongly coupled parabolic system from incompressible miscible flow in porous media with a commonly-used Galerkin-mixed FEM and linearized semi-implicit Euler scheme. Optimal  $L^2$  error estimates were obtained almost without any time-step (convergence) condition, while all previous works require certain restriction for time-step size. We believe that the idea of the error splitting coupled with the regularity analysis of the time-discrete PDEs can be applied to many other nonlinear parabolic PDEs and time discretizations to obtain optimal error estimates unconditionally.

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